Rec. #4

SOLUTIONS

1) Right hand derivative of $f(x)$ at $x=1$ $\qquad f(1) = 0^2 + \sin 0 = 0$

$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{|1+h-1|(1+h)^2 + \sin(1+h-1) - 0}{h}$$

$$= \lim_{h \to 0^+} \frac{h(1+h)^2 + \sin h}{h} = \lim_{h \to 0^+} (1+h)^2 + \frac{\sin h}{h} = 2$$

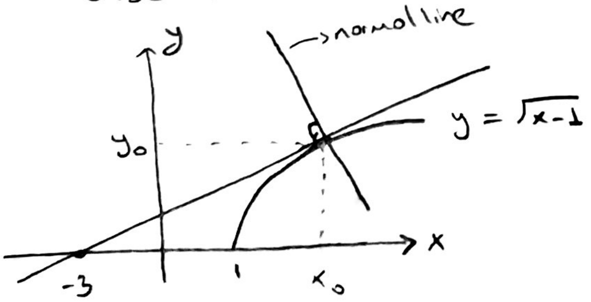
Left hand derivative of $f(x)$ at $x=1$,

$$\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} \frac{|1+h-1|(1+h)^2 + \sin(1+h-1) - 0}{h}$$

$$= \lim_{h \to 0^-} \frac{-h(1+h)^2 + \sin h}{h} = \lim_{h \to 0} -(1+h)^2 + \frac{\sin h}{h} = 0$$

Since right and left derivative of $f(x)$ at $x=1$ does not equal, f is <u>not</u> differentiable at $x=1$, $f'(1)$ does not exist.

2) Suppose that the point $P(x_0, y_0)$ where the tangent line to the curve $y = \sqrt{x-1}$ crosses the x-axis at $x=-3$. $P(x_0, y_0)$ is also on the curve, that is $y_0 = \sqrt{x_0 - 1}$.



* We have to find the slope of tangent line (M_T)
* We have to find the equation of tangent line
* We have to find (x_0, y_0)

$$M_T = \left.\frac{df}{dx}\right|_{x=x_0} = \frac{1}{2\sqrt{x_0-1}}, \quad (x_0, y_0) \quad \text{and} \quad y_0 = \sqrt{x_0-1}.$$

Equation of the tangent line: $\quad y = y_0 + \dfrac{1}{2\sqrt{x_0-1}} \cdot (x - x_0)$

The tangent line passes through $(-3,0)$, so

$$0 = y_0 + \frac{1}{2\sqrt{x_0-1}}(-3-x_0) \quad , \quad \text{substitue } \sqrt{x_0-1} \text{ on } y_0$$

$$\frac{3+x_0}{2\sqrt{x_0-1}} = \sqrt{x_0-1} \Rightarrow 3+x_0 = 2(x_0-1)$$

$$\Rightarrow x_0 = 5 \Rightarrow y_0 = \sqrt{5-1} = 2$$

$$\Rightarrow m_t = \frac{1}{2\cdot\sqrt{5-1}} = \frac{1}{4}$$

Hence, the equation of the tangent line

$$y = 2 + \frac{1}{4}(x-5) \Rightarrow y = \frac{1}{4}x + \frac{3}{4}$$

The slope of the normal line: $m_N = -\frac{1}{m_T}$, so $m_N = -4$

and $P(5,2)$,

The equation of the normal line,

$$y = 2 - 4(x-5) \Rightarrow y = -4x + 22 \;//$$

3) The slope of the horizontal tangent at the point $x = x_0$ is $m = 0$.

$f(x) = x^2 + 4x - 1$

$f'(x) = 2x + 4$

$m = f'(x_0) = 2x_0 + 4$

$2x_0 + 4 = 0 \Rightarrow x_0 = -2$

Hence, the function $f(x)$ has horizontal tangent at $x=-2$ //

4) $\operatorname{cosec}(x^2+y^2) = 1 \Rightarrow \dfrac{1}{\sin(x^2+y^2)} = 1 \Rightarrow \sin(x^2+y^2) = 1$

So, $x^2+y^2 = \frac{\pi}{2} + 2k\pi \quad , \quad k \in \mathbb{Z}$

By using implicit differentiation, we get

$$2x + 2 . y . y' = 0 \Rightarrow y' = -\frac{x}{y}$$

We may have vertical, tangent $y=0$. Then

$$x^2 + 0^2 = \frac{\pi}{2} + 2k\pi$$

$$x = \mp\sqrt{\frac{\pi}{2} + 2k\pi} \; .$$

5) a. $y = \dfrac{x^3+7}{x} \Rightarrow y = x^2 + 7.x^{-1} \Rightarrow y' = 2x - 7x^{-2}$

b. $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi+1} \Rightarrow y' = 7x^6 + \sqrt{7}$

c. $y = (2x-5)(4-x)^{-1} \Rightarrow y' = (2x-5)'(4-x)^{-1} + (2x-5)[(4-x)^{-1}]'$

Using product rule $\Rightarrow y' = 2(4-x)^{-1} + (2x-5)(-1(4-x)^{-2}(-1))$

$$\Rightarrow y' = \frac{2}{4-x} + \frac{2x-5}{(4-x)^2} = \frac{2(4-x)+2x-5}{(4-x)^2} = \frac{3}{(4-x)^2}$$

Using quotient rule $y = \dfrac{2x-5}{4-x} \Rightarrow y' = \dfrac{(2x-5)'(4-x) - (2x-5)(4-x)'}{(4-x)^2}$

$$y' = \frac{2(4-x) - (2x-5)(-1)}{(4-x)^2} = \frac{3}{(4-x)^2}$$

d. $y = \dfrac{x(x+1)(x^2-x+1)}{x^4} = \dfrac{x^3-1}{x^3} = 1 - x^{-3}$

$$\Rightarrow y' = -(-3)x^{-4} \Rightarrow y' = 3/x^4$$

e. $y = (x^2+1)\left(x+5+\frac{1}{x}\right)$

$$y' = (x^2+1)'\left(x+5+\frac{1}{x}\right) + (x^2+1)\left(x+5+\frac{1}{x}\right)'$$

$$= 2x\left(x+5+\frac{1}{x}\right) + (x^2+1)\left(1-\frac{1}{x^2}\right)$$

$$= 2x^2+10x+2 + \frac{x^4-1}{x^2} = 3x^2+10x+2-x^{-2} \,//$$

f. $y = (\sec x + \tan x)(\sec x - \tan x) = \sec^2 x - \tan^2 x = 1$, since $1+\tan^2 x = \sec^2 x$

So, $y = 1 \Rightarrow y' = 0$

OR $y = \sec^2 x - \tan^2 x$

$$y' = 2\sec x . \underbrace{\sec x . \tan x}_{(\sec x)'} - 2\tan x . \underbrace{\sec^2 x}_{(\tan x)'} = 0 \,//$$

g. $y = \tan(x + \cos x)$, $y' = \frac{dy}{dx} = ?$

$\{$ Chain rule or outside-inside rule, $y = f(g(x)) \Rightarrow y' = f'(g(x)) . g'(x) \}$

Let $u = x + \cos x$, then $y = \tan u$

$u = x + \cos x$

By using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \sec^2 u . (1-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2(x+\cos x)(1-\sin x)$$

h) $y = \tan^2(\sin^3 x)$, $\frac{dy}{dx} = ?$

Let $y = u^2$

$u = \tan(v)$

$v = w^3$

$w = \sin x$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dw} . \frac{dw}{dx}$$

$$= 2u . \sec^2(v) . 3w^2 . \cos x$$

$$= 2\tan(\sin^3 x) . \sec^2(\sin^3 x) . 3\sin^2 x . \cos x \,//$$

i') $y = \sec\sqrt{x} \cdot \tan\frac{1}{x}$

$y' = (\sec\sqrt{x})' \cdot \tan\frac{1}{x} + \sec\sqrt{x} \cdot \left(\tan\frac{1}{x}\right)'$

$= \sec\sqrt{x} \cdot \tan\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot \tan\frac{1}{x} + \sec\sqrt{x} \cdot \sec^2\frac{1}{x} \cdot \frac{-1}{x^2}$

6) $y = \cot\left(\frac{\sin x}{x}\right)$

quotient rule

$y' = -\csc^2\left(\frac{\sin x}{x}\right) \cdot \left(\frac{\sin x}{x}\right)' = -\csc^2\left(\frac{\sin x}{x}\right) \cdot \frac{(\cos x)\, x - \sin x}{x^2}$

b) $y = \left(\frac{\sin x}{1+\cos x}\right)^2$

$y' = 2 \cdot \left(\frac{\sin x}{1+\cos x}\right) \cdot \left(\frac{\sin x}{1+\cos x}\right)'$

$= 2 \cdot \left(\frac{\sin x}{1+\cos x}\right) \cdot \frac{\cos x\,(1+\cos x) - \sin x\,(-\sin x)}{(1+\cos x)^2} = \frac{2\sin x}{(1+\cos x)^2}$

c) $y = x^{-3}\sec^2(2x)$

$y' = -3 \cdot (x^{-4}) \cdot \sec^2(2x) + x^{-3} \cdot 2\sec(2x) \cdot \underbrace{(\sec 2x)(\tan 2x) \cdot 2}_{(\sec 2x)'}$

d) $y = \frac{\tan x}{1+\tan x} \Rightarrow y' = \frac{\sec^2 x\,(1+\tan x) - (\tan x)(\sec^2 x)}{(1+\tan x)^2} = \frac{\sec^2 x}{(1+\tan x)^2}$

e) $y = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)^2$

$y' = 2\left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)\left(\frac{\csc\theta\cot\theta}{2} - \frac{\theta}{2}\right)$

f) $y' = 4(1-x)^3(-1)(1+\sin^2 x)^{-5} + (1-x)^4 \cdot -5(1+\sin^2 x)^{-6} \cdot 2\sin x.\cos x$

7) $y' = 6x^2 - 6x - 12$

a) perpendicular to the line $y = 1 - \frac{x}{24}$, slope $= -\frac{1}{24}$

the slope of tangent line $m_T = 24$ since orthogonality

$6x^2 - 6x - 12 = 24 \Rightarrow x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$ $x = 3$ or $x = -2$

b) parallel to the line $y = \sqrt{2} - 12x$, slope $= -12$

the slope of tangent line $m_T = -12$

$6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$ $x = 0$,

or $x = 1$,

Q: Use implicit differentiation to find $\frac{dy}{dx}$.

a) $2xy + y^2 = x + y$ b) $y^2 \cdot \cos\left(\frac{1}{y}\right) = 2x + 2y$

Solution a) $2xy + y^2 = x + y$, We cannot solve the equation for y as an explicit function of x,

So, we must use implicit differentiation.

$$\frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(x) + \frac{d}{dx}(y)$$

$$2.1.y + 2x.\frac{dy}{dx} + 2.y.\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2y + 2x.y' + 2y.y' = 1 + y'$$

$$y'(2x + 2y - 1) = 1 - 2y$$

$$y' = \frac{1 - 2y}{2x + 2y - 1} \quad //$$

b) $$\frac{d}{dx}\left(y^2 \cdot \cos\left(\frac{1}{y}\right)\right) = \frac{d}{dx}(2x + 2y)$$

$$\frac{d}{dx}(y^2) \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \frac{d}{dx}\left(\cos\left(\frac{1}{y}\right)\right) = 2 + 2.\frac{dy}{dx}$$

$$2.y.\frac{dy}{dx} \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \left[-\sin\left(\frac{1}{y}\right) \cdot \frac{-1}{y^2} \cdot \frac{dy}{dx}\right] = 2 + 2.\frac{dy}{dx}$$

$$2y \cdot y' \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \sin\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} \cdot y' - 2y' = 2$$

$$y' = \frac{2}{2y \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \sin\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} - 2}$$

Q: Find the value of $\frac{d^2y}{dx^2}$ for the following function

$xy + y^2 = 1$ at the point $(0, -1)$.

Solution:

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(1)$$

$$1.y + x.y' + 2yy' = 0 \Rightarrow y' = \frac{-y}{x+2y}$$

Take second derivative,

$$y' + 1.y' + x.y'' + 2y'.y' + 2y.y'' = 0$$

$$y''(x+2y) = -2y' - 2(y')^2$$

$$y'' = \frac{-2y' - 2(y')^2}{x+2y}$$

$$y'' = \frac{-2\left(\frac{-y}{x+2y}\right) - 2\left(\frac{-y}{x+2y}\right)^2}{x+2y}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(0,-1)} = \frac{-2\left(\frac{1}{-2}\right) - 2\left(\frac{1}{-2}\right)^2}{-2}$$

$$= \frac{+1 - \frac{1}{2}}{-2} = -\frac{1}{4}$$

Q: Find all the points on the curve which have ~~m~~ slope -1

$$x^2 y^2 + xy = 2$$

Solution: We need to find the points (x_0, y_0) such that $y'\big|_{(x_0,y_0)} = -1$

To find y', we'll use implicit differentation, because we cannot solve the equation for y as an explicit function of x.

$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(2)$$

$$2x.y^2 + x^2.2.y.y' + 1.y + xy' = 0$$

$$2x_0.y_0^2 + x_0^2\, 2y_0.(-1) + y_0 + x_0(-1) = 0$$

$$2x_0y_0^2 - 2x_0^2y_0 + y_0 - x_0 = 0.$$

$$2x_0y_0(y_0 - x_0) + (y_0 - x_0) = 0.$$

$$(2x_0y_0 + 1)(y_0 - x_0) = 0 \Rightarrow y_0 = x_0 \quad \text{or} \quad x_0y_0 = \frac{-1}{2}$$

- For $x_0 = y_0$,

$$x_0^2x_0^2 + x_0x_0 = 2$$

$$x_0^4 + x_0^2 - 2 = 0$$

$$(x_0^2+2)(x_0^2-1) = 0 \Rightarrow x_0^2 = 1 \quad x_0 = 1, \quad x_0 = -1$$

$$x_0^2 \neq -2$$

Thus, $(1,1)$ or $(-1,-1)$

- For $x_0y_0 = -\frac{1}{2}$, $\left(-\frac{1}{2}\right)^2 - \frac{1}{2} \neq 2$

Q: Find an equation for the tangent line to each of the following parametrized curves at the given value. Also find the value of $\frac{d^2y}{dx^2}$ at the given point.

a) $x = \sec^2 t - 1$, $y = \tan t$, $t = -\frac{\pi}{4}$

b) $x = -\sqrt{t+1}$, $y = \sqrt{3t}$, $t = 3$.

Solution: a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{2\sec t \cdot \sec t \cdot \tan t} = \frac{1}{2\tan t}$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}} = \frac{1}{2\tan(-\frac{\pi}{4})} = -\frac{1}{2} \, // \quad \text{(slope of tangent line)}$$

if $t = -\frac{\pi}{4}$, then $x = \sec^2(-\frac{\pi}{4}) - 1 = \frac{1}{\cos^2(-\frac{\pi}{4})} - 1 = 1$

$$y = \tan(-\frac{\pi}{4}) = -1$$

$m = -\frac{1}{2}$ $\qquad (x_0, y_0) = (1, -1)$

The equation of tangent line is that $y = -1 + (-\frac{1}{2})(x-1)$

$$2y = -1 - x \, //$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left[(2\tan t)^{-1}\right]'}{2\sec^2 t \cdot \tan t} = \frac{-1(2\tan t)^{-2} \cdot 2\sec^2 t}{2\sec^2 t \cdot \tan t}$$

$$= -\frac{1}{4\tan^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = \frac{1}{4} \, //$$

b) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{3t}}\cdot 3}{-\frac{1}{2\sqrt{t+1}}} = \frac{-3\sqrt{t+1}}{\sqrt{3t}} = -3\sqrt{\frac{t+1}{3t}}$$

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{-3\cdot 2}{3} = -2 \;//$$

when $t=3$, $x = -\sqrt{3+1} = -2$ $\quad (-2, 3)$

$y = \sqrt{3\cdot 3} = 3$

The equ. of tangent line: $y = 3 + (-2)(x+2)$

$y = -1 - 2x \;//$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left[-3\cdot\left(\frac{t+1}{3t}\right)^{1/2}\right]'}{-\frac{1}{2\sqrt{t+1}}} = 6\sqrt{t+1}\cdot\left[\left(\frac{t+1}{3t}\right)^{1/2}\right]'$$

$$= 6\sqrt{t+1}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{t+1}{3t}}}\cdot\frac{3t-(t+1)3}{9t^2}$$

$$= 6\cdot\frac{1}{2}\cdot 3\cdot\frac{1}{9}\cdot\sqrt{t+1}\cdot\frac{\sqrt{3t}}{\sqrt{t+1}}\cdot\frac{-1}{t^2}$$

$$= -\frac{\sqrt{3t}}{t^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=3} = -\frac{3}{9} = -1/3 \;//$$

Q: Assuming that the following equations define x and y implicitly as differentiable functions $x = f(t)$ and $y = g(t)$.

$$x \sin t + \sqrt{x} = t \quad , \quad t \sin t - 2t = y \quad , \quad t = \pi$$

Find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t. and write the tangent line at $t = \pi$

Solution: We'll calculate the value of $\left.\frac{dy}{dx}\right|_{t=\pi}$

Slope: $$\left.\frac{dy}{dx}\right|_{t=\pi} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi}$$

$$\left.\frac{dy}{dt}\right|_{t=\pi} = (\sin t + t\cos t - 2)\Big|_{t=\pi} = \sin\pi + \pi.\cos\pi - 2 = -\pi - 2$$

$$\left.\frac{dx}{dt}\right|_{t=\pi} = ?$$

$$\frac{d}{dt}(x.\sin t) + \frac{d}{dt}(\sqrt{x}) = \frac{d}{dt}(t)$$

$$\frac{dx}{dt}.\sin t + x.\underbrace{\frac{d}{dt}(\sin t)}_{\cos t} + \frac{1}{2\sqrt{x}}.\frac{dx}{dt} = 1$$

$$\frac{dx}{dt} = \frac{1 - x.\cos t}{\sin t + \frac{1}{2\sqrt{x}}}$$

when $t = \pi$, $x.\sin\pi + \sqrt{x} = \pi \Rightarrow x = \pi^2$

$\pi.\sin\pi - 2\pi = y \Rightarrow y = -2\pi$

$$\left.\frac{dx}{dt}\right|_{t=\pi} = \frac{1 - \pi^2.\cos\pi}{\sin\pi + \frac{1}{2\sqrt{\pi^2}}} = \frac{1-\pi^2}{\frac{1}{2\pi}} = 2\pi - 2\pi^3$$

$$\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{-\pi-2}{2\pi-2\pi^3}$$

Tangent line:

$$y = -2\pi + \left(\frac{-\pi-2}{2\pi-2\pi^3}\right)(x-\pi^2)$$

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Q: Find the $\frac{dy}{dx}$ for the following functions

chain rule
outside-inside rule
$y = f(g(x))$
$y' = f'(g(x)) . g'(x)$

a) $y = \tan(x + \cos x)$

b) $y = \tan^2(\sin^3 x)$

c) $y = \cos\left(5 \sin \frac{x}{3}\right)$

Solution

a) $y = \tan u$

$u = x + \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u . (1 - \sin x)$$

$$= \sec^2(x + \cos x) . (1 - \sin x)$$

b) $y = u^2$

$u = \tan v$

$v = t^3$

$t = \sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdot \frac{dt}{dx}$$

$$= 2u . \sec^2 v . 3t^2 . \cos x$$

$$= 2 . \tan(\sin^3 x) . \sec^2(\sin^3 x) . 3 . \sin^2 x . \cos x$$

$$y' = 2 \tan(\sin^3 x) . [\tan(\sin^3 x)]'$$

$$= 2 \tan(\sin^3 x) . \sec^2(\sin^3 x) . (\sin^3 x)'$$

$$= 2 \tan(\sin^3 x) . \sec^2(\sin^3 x) . 3 \sin^2 x . \cos x$$

c) $y = \cos u$

$u = 5 \sin v$

$v = \frac{x}{3}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= -\sin u . 5 \cos v . \frac{1}{3}$$

$$= -\sin\left(5 \sin \frac{x}{3}\right) . 5 \cos \frac{x}{3} \cdot \frac{1}{3}$$

Q: Find $\dfrac{d^2y}{dx^2}$ for the following implicit function $y^2 = 1 - \dfrac{2}{x}$

~~b) Evaluate $\frac{d^2y}{dx^2}$ at the point (~~

Solution: $\dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(1 - 2.x^{-1})$

$$2.y.y' = (-2)(-1)\,x^{-2}$$

$$2y.y' = 2x^{-2} \Rightarrow y' = \frac{1}{x^2.y}$$

Take second derivative.

$$\frac{d}{dx}(2y.y') = \frac{d}{dx}(2x^{-2})$$

$$\frac{d}{dx}(2y).y' + 2y.\frac{d}{dx}(y') = 2(-2)\,x^{-3}$$

$$2y'.y' + 2y.y'' = -4x^{-3}$$

$$y'' = \frac{-4x^{-3} - 2(y')^2}{2y}$$

$$y'' = \frac{-4x^{-3} - 2(x^{-2}.y^{-1})^2}{2y} = \frac{-4x^{-3} - 2x^{-4}y^{-2}}{2y}$$

$$= -\frac{2xy^2 + 1}{x^4y^3}$$

//

$|\sin b - \sin a| \leq |b-a|$

Solution:

Let $a, b \in \mathbb{R}$ be given arbitrarily with $a<b$.

Consider the function $f(x) = \sin x$ which is continuous and differentiable everywhere.

- $f(x) = \sin x$ is continuous on $[a,b]$
- $f(x) = \sin x$ is differentiable on (a,b)

By the Mean Value Theorem,

There exists at least one $c \in (a,b)$ s.t $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Then $|f'(c)| = \dfrac{|f(b)-f(a)|}{|b-a|}$

Since $f'(x) = \cos x$, $f'(c) = \cos c$ and so $0 \leq |f'(c)| \leq 1$

Therefore, $\dfrac{|f(b)-f(a)|}{|b-a|} \leq 1$. That is

$|\sin b - \sin a| \leq |b-a|$ //

$\square$.

Prove that the functions $f(x) = \dfrac{x}{x^4+1}$ and $g(x) = \dfrac{x}{x^3+1}$ satisfy the equation $f'(x) = g'(x)$ at least one x in the interval $(0,1)$.

<u>solution:</u> Let $H(x) = f(x) - g(x)$.

$$H(x) = \frac{x}{x^4+1} - \frac{x}{x^3+1} = \frac{(1-x)x^4}{(1+x^4)(1+x^3)}$$

$$H'(x) = f'(x) - g'(x)$$

Also, $H(0) = f(0) - g(0) = 0$ and

$H(1) = f(1) - g(1) = 0$

The function $H(x)$ satisfies the hypotheses of Rolle's Theorem on $[0,1]$. It is continuous on $[0,1]$ and differentiable on $(0,1)$ since both $f(x)$ and $g(x)$ are.

Therefore, $H'(c) = 0$ at some point $c \in (0,1)$

$$H'(c) = f'(c) - g'(c) \Rightarrow f'(c) = g'(c).$$

<u>Rolle Theorem</u>

- $f(x)$ is continuous on the closed interval $[a,b]$
- $f(x)$ is differentiable on the open interval (a,b)
- $f(a) = f(b)$

$\Rightarrow \exists c \in (a,b)$ such that $f'(c) = 0$ //

<u>Intermediate Value Theorem</u>

$$\left.\begin{array}{l} \bullet\ f(x) \text{ is continuous on the closed interval } [a,b] \\ \bullet\ M \text{ is any number between } f(a) \text{ and } f(b) \end{array}\right\} \Rightarrow \begin{array}{l} \exists c \in (a,b) \text{ s.t.} \\ f(c) = M \end{array}$$

Show that $2x^3+x+4=0$ has exactly one zero.

solution: Define $f(x) = 2x^3+x+4$.

(i) $f(-2) = 2.(-2)^3-2+4 = -14 < 0$

$f(0) = 4 > 0$

By Intermediate Value Theorem, we get that $f(x)$ which is a continuous func. has at least one zero between -2 and 0.

(ii) Suppose that $f(x)$ has more than one zero. Let these zeros be $x=a$ and $x=b$, $a<b$.

Apply Rolle' Theorem on the interval $[a,b]$.

- $f(x)$ is continuous
- $f(x)$ is diffentiable, because

$f'(x) = 6x^2+1 \quad , \forall x \in (-\infty,\infty)$

- $f(a) = f(b) = 0$

Therefore, by Rolle Theorem, $\exists c \in (a,b) : f'(c)=0$,

but $f'(x) = 6x^2+1 > 0$ for $x \in (-\infty,\infty)$. $(f(x) \nearrow)$

f can have at most one zero on $\mathbb{R}$.

(i) $\wedge$ (ii) $\Rightarrow$ f has exactly one zero on $\mathbb{R}$.

Does the fuction $f(x) = \sqrt{x - x^2}$ satisfy the hypothesis of Mean Value Theorem on the interval $[0,1]$? If so, find the admissible value of c.

Solution :

Recall: Mean Value Theorem

- $f(x)$ is continuous on the closed interval $[a,b]$
- $f(x)$ is differentiable on the open interval (a,b)

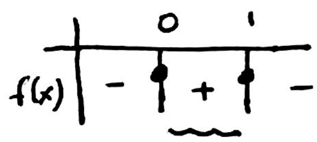
$\Rightarrow$ There is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

— — — — — — —

- Continuity:

- $f(x) = \sqrt{x - x^2}$ $\qquad x - x^2 \geq 0 \Rightarrow x(1-x) \geq 0$



$f(x) \geq 0$

Domain of f: $[0,1]$

So, $f(x)$ is continuous on $[0,1]$.

- Differentiability:

$$f'(x) = \frac{1}{2\sqrt{x - x^2}} \cdot (1 - 2x) = \frac{1 - 2x}{2\sqrt{x - x^2}}$$

$f'(x)$ is defined on $(0,1)$. Hence f is differentiable on $(0,1)$.

Therefore, MVT's hypothesis are satisfied.

From MVT, we can say that

$\exists c \in (0,1)$ such that $f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = 0$

$$f'(c) = \frac{1 - 2c}{2\sqrt{c - c^2}} = 0 \Rightarrow c = 1/2 \,//$$

$\square$.

Q: Sketch the graph of $f(x) = \dfrac{x^3}{x^2-9}$

Solution:

① Domain: $x^2-9=0 \quad x=3 \quad x=-3$. So $D_f: \mathbb{R}-\{-3,3\}$

② Intercepts: $x=0 \Rightarrow y=0 \qquad (0,0)$

$y=0 \Rightarrow x=0$

③ Symmetry: $f(-x) = \dfrac{-x^3}{x^2-9} = -f(x)$, f is odd

So, its graph is symmetric about the origin.

④ Asymptotes:

Horizontal Asymptote:

$$\lim_{x\to\infty} \frac{x^3}{x^2-9} = \lim_{x\to\infty} \frac{\cancel{x^2}\cdot x}{\cancel{x^2}(1-9/x^2)} = \infty$$

$$\lim_{x\to-\infty} \frac{x^3}{x^2-9} = -\infty$$

There is no horizontal asymptote.

Vertical Asymptote: f is undefined at $x=3$ and $x=-3$

$$\lim_{x\to3^+} \frac{x^3}{x^2-9} = \frac{27}{0^+} = +\infty \qquad \lim_{x\to-3^+} \frac{x^3}{x^2-9} = \frac{-27}{0^-} = +\infty$$

$$\lim_{x\to3^-} \frac{x^3}{x^2-9} = \frac{27}{0^-} = -\infty \qquad \lim_{x\to-3^-} \frac{x^3}{x^2-9} = \frac{-27}{0^+} = -\infty$$

$x=3$ is vertical asymptote $\qquad x=-3$ is vertical asymptote.

Oblique Asymptote: Let $y=mx+n$ be oblique asymptote of $f(x)$.

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3}{x^3-9x} = 1$$

$$n = \lim_{x\to\infty} [f(x)-mx] = \lim_{x\to\infty} \frac{x^3}{x^2-9} - x = \lim_{x\to\infty} \frac{9x}{x^2-9} = 0$$

So, $y=x$ is an oblique asymptot.

⑤ Extremum Points, Increasing and Decreasing

$$f'(x) = \frac{3x^2(x^2-9) - x^3\, 2x}{(x^2-9)^2} = \frac{x^2(x^2-27)}{(x^2-9)^2}$$

- $f'(x) = 0$ when $x = 0$, $x = 3\sqrt{3}$ and $x = -3\sqrt{3}$ $\in D_f$

These are critical points.

$f(0) = 0$ $\quad f(3\sqrt{3}) = \frac{81\sqrt{3}}{18} = \frac{9\sqrt{3}}{2}$ $\quad f(-3\sqrt{3}) = -\frac{9\sqrt{3}}{2}$

- f' undefined at $x = 3$, $x = -3$ $\notin D_f$

		$-3\sqrt{3}$		-3		0		3		$3\sqrt{3}$	
$f'(x)$	$+$	0	$-$	‖	$-$	0	$-$	‖	$-$	0	$+$
$f(x)$	↗		↘	‖	↘		↘	‖	↘		↗

f' changes sign at $x = -3\sqrt{3}$, $3\sqrt{3}$
$\Rightarrow$ there is local max
min.

f is increasing on $(-\infty, -3\sqrt{3}] \cup [3\sqrt{3}, \infty)$

f is decreasing on $[-3\sqrt{3}, 3\sqrt{3}] - \{-3, 3\}$

f has local maximum at $-3\sqrt{3}$.

f has local minimum at $3\sqrt{3}$.

Since f' does not change sign at 0, so there is no local maximum or minimum.

⑥ Concavity, Inflection points

$$f''(x) = \frac{18x(x^2+27)}{(x^2-9)^3}$$

. $f''(x) = 0$ when $x = 0$

. $f''(x)$ undefined at $x = 3, -3 \notin D_f$

x		-3		0		3	
f''	$-$	‖	$+$	0	$-$	‖	$+$
$f(x)$	$\cap$	‖	$\cup$		$\cap$	‖	$\cup$

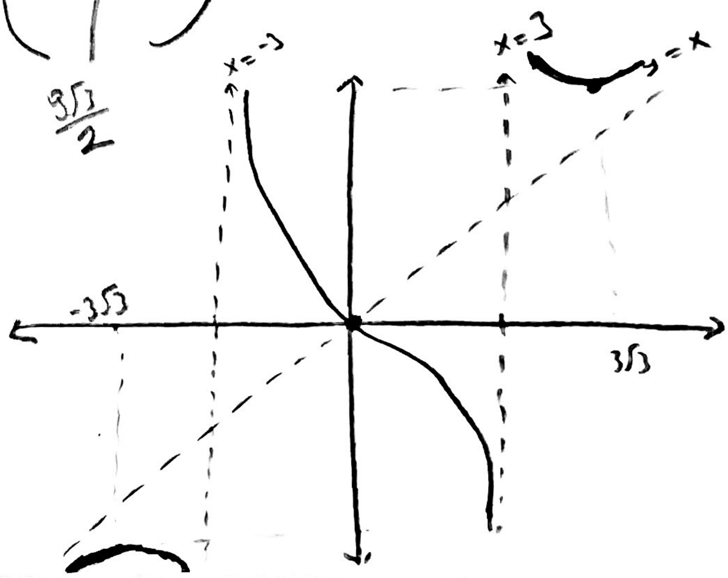
f has an inflection point at 0 because f'' change sing at 0.

⑦ Table & Graph.

x		$-3\sqrt{3}$		-3		0		3		$3\sqrt{3}$	
f'	↗		↘	‖	↘		↘	‖	↘		↗
f''	$\cap$		$\cap$	‖	$\cup$		$\cap$	‖	$\cup$		$\cup$
f		$-\frac{9\sqrt{3}}{2}$	$-\infty$	‖	$+\infty$	0	$-\infty$	‖	$+\infty$	$\frac{9\sqrt{3}}{2}$	

$x = 3, -3$ vertical

$y = x$ oblique



Soru: $f(x)=\dfrac{x^3-4x}{x^2-1}$ fonksiyonunu grafiğini çiziniz.

① Tanım Aralığı:

$x=\mp 1$'de $f(x)$ tanımsızdır. T.A: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

② Eksenleri Kesen Noktaları:

$x=0 \Rightarrow y=f(0)=\dfrac{0-0}{0-1}=0 \Rightarrow (0,0)$

$y=0 \Rightarrow 0=\dfrac{x^3-4x}{x^2-1} \Rightarrow x(x^2-4)=0 \Rightarrow x=0 \quad x=\mp 2$ $\Big\} \Rightarrow (0,0)\ (2,0)\ (-2,0)$

③ Simetri:

$$f(-x)=\frac{(-x)^3-4(-x)}{(-x)^2-1}=-\frac{x^3-4x}{x^2-1}=-f(x)$$

$\Rightarrow f(x)$ tek fonksiyondur. $f(x)$'in grafiği orijine göre simetriktir

④ Asimptotlar

Yatay asimptot:

$$\lim_{x\to\infty}\frac{x^3-4x}{x^2-1}=\lim_{x\to\infty}\frac{x^2(x-4/x)}{x^2(1-1/x^2)}=+\infty$$

$$\lim_{x\to-\infty}\frac{x^3-4x}{x^2-1}=-\infty$$

Yatay Asimptot yoktur.

Düşey Asimptot:

$$\lim_{x\to1^+}\frac{x^3-4x}{x^2-1}=\frac{-3}{0^+}=-\infty \qquad \lim_{x\to-1^+}\frac{x^3-4x}{x^2-1}=\frac{3}{0^-}=-\infty$$

$$\lim_{x\to1^-}\frac{x^3-4x}{x^2-1}=\frac{-3}{0^-}=+\infty \qquad \lim_{x\to-1^-}\frac{x^3-4x}{x^2-1}=\frac{3}{0^+}=+\infty$$

$\Rightarrow x=1$ düşey asimptot

$\Rightarrow x=-1$ düşey asimptot.

Eğik asimptot:

$$\begin{array}{r|l} x^3 - 4x & x^2 - 1 \\ \cline{2-2} -\;\; x^3 - x & x \\ \hline -3x & \end{array}$$

$$\Rightarrow f(x) = \frac{x^3 - 4x}{x^2 - 1} = \boxed{x} + \frac{-3x}{x^2 - 1}$$

$\Rightarrow y = x$ eğik asimptottur.

⑤ Ekstremum Noktaları, Artanlık ve Azalanlık

$$f'(x) = \frac{(3x^2 - 4)(x^2 - 1) - (x^3 - 4x)(2x)}{(x^2 - 1)^2} = \frac{x^4 + x^2 + 4}{(x^2 - 1)^2}$$

Her x için $f'(x) > 0$. $x = \mp 1$ için f' tanımsızdır.

Fakat $x = \mp 1 \notin T.A$

Dolayısıyla kritik nokta yok.

x		-1		1	
$f'(x)$	$+$	‖	$+$	‖	$+$
$f(x)$	↗	‖	↗	‖	↗

$\forall x \in T.A$, $f(x)$ artandır.

⑥ Konkavlık ve Büküm Noktaları

$$f''(x) = \frac{(4x^3 + 2x)(x^2 - 1)^2 - (x^4 + x^2 + 4).2.(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{-6x(x^2 + 3)}{(x^2 - 1)^3}$$

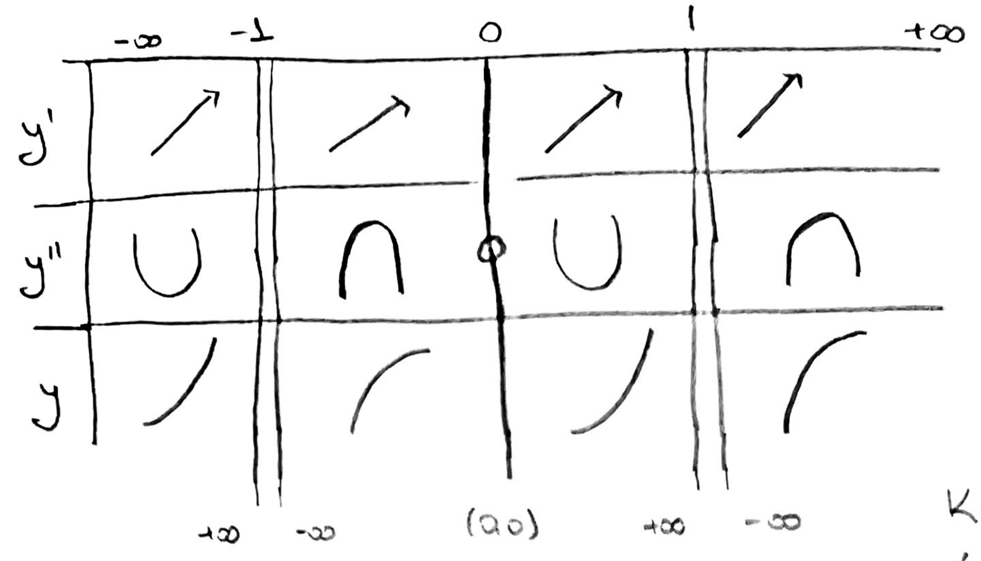
$f''(x) = 0 \Rightarrow -6x(x^2 + 3) = 0 \Rightarrow x = 0 \in T.A$ Aday büküm noktası

$f''(x)$ tanımsız olduğu noktalar: $x = \mp 1 \notin T.A$

x		-1		0		1	
$f''(x)$	$+$	‖	$-$	0	$+$	‖	$-$
$f(x)$	∪	‖	∩		∪	‖	∩

Büküm Noktası
(0,0)

⑦ Özet Tablo & Grafik



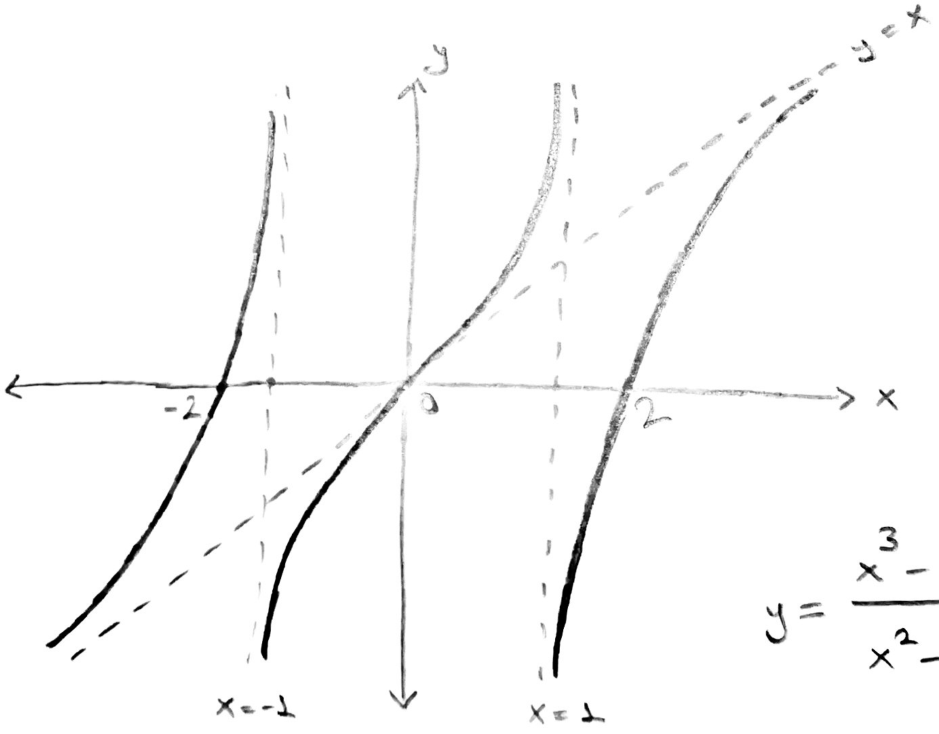
Büküm Noktası

Kesenler:

(0,0) (2,0) (-2,0)

Eğik Asimptot

$y = x$



$$y = \frac{x^3 - 4x}{x^2 - 1}$$ 'in grafiği.

Soru: $f(x) = \frac{(x-1)^2}{x+2}$ fonksiyonun grafiğini çiziniz.

① Tanım Aralığı:

$x = -2$ 'de f tanımsızdır. T.A: $(-\infty, -2) \cup (-2, \infty)$

② Eksenleri Kesen Noktalar:

$x = 0 \Rightarrow y = f(0) = \frac{1}{2} \Rightarrow (0, \frac{1}{2})$

$y = 0 \Rightarrow 0 = \frac{(x-1)^2}{x+2} \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$ $(1, 0)$

③ Simetri

$f(-x) = \frac{(-x-1)^2}{-x+2} = \frac{(x+1)^2}{-x+2} \Rightarrow$ $f(-x) \neq f(x)$, $f(-x) \neq -f(x)$

$f(x)$, ne tek ne de çift fonksiyondur. Simetri yoktur.

④ Asimptotlar

Yatay asimptot:

$\lim_{x\to\infty} \frac{(x-1)^2}{x+2} = \lim_{x\to\infty} \frac{x^2-2x+1}{x+2} = \frac{\cancel{x}(x-2+1/x)}{\cancel{x}(1+2/x)} = \infty$

$\lim_{x\to-\infty} \frac{(x-1)^2}{x+2} = -\infty$ Yatay asimptot yoktur

Düşey asimptot:

$\lim_{x\to -2^+} \frac{(x-1)^2}{x+2} = \frac{9}{0^+} = \infty$

$\lim_{x\to -2^-} \frac{(x-1)^2}{x+2} = \frac{9}{0^-} = -\infty$

$x = -2$ 'de düşey asimptot vardır.

Eğik Asimptot:

$(x-1)^2 = x^2 - 2x + 1 \Rightarrow$

$$\begin{array}{r|l} x^2-2x+1 & x+2 \\ -(x^2+2x) & x-4 \\ \hline -4x+1 & \\ -(-4x-8) & \\ \hline 9 & \end{array}$$

$\Rightarrow f(x) = \boxed{x-4} + \frac{9}{x+2}$

$y = x-4$ eğik asimptot vardır

⑤ Ekstremum Noktaları, Artanlık ve Azalanlık

$$f'(x) = \frac{2(x-1)(x+2) - (x-1)^2(1)}{(x+2)^2} = \frac{(x-1)(x+5)}{(x+2)^2}$$

$f'(x) = 0 \Rightarrow x = 1 \in T.A$ ve $x = -5 \in T.A$

f' tanımsız $\Rightarrow x = -2 \notin T.A$

Kritik noktalar: $x = 1 \Rightarrow f(1) = 0$ $(1, 0)$

$x = -5 \Rightarrow f(-5) = \frac{36}{-3} = -12$ $(-5, -12)$

x		-5		-2		1	
$f'(x)$	$+$	0	$-$	‖	$-$	0	$+$
$f(x)$	↗		↘	‖	↘		↗

$(-5,-12)$ Yerel Maks.

$(1,0)$ Yerel Min.

Azalan old. aralık: $[-5, 1] - \{-2\}$

Artan old. aralık: $(-\infty, -5] \cup [1, \infty)$

⑥ Konkavlık ve Büküm Noktaları

$$f''(x) = \left[\frac{(x-1)(x+5)}{(x+2)^2}\right]' = \left[\frac{x^2+4x-5}{(x+2)^2}\right]' = \frac{(2x+4)(x+2)^2 - (x^2+4x-5)2(x+2)}{(x+2)^4}$$

$$= \frac{(2x+4)(x+2) - (x^2+4x-5).2}{(x+2)^3} = \frac{2x^2+8x+8-2x^2-8x+10}{(x+2)^3} = \frac{18}{(x+2)^3}$$

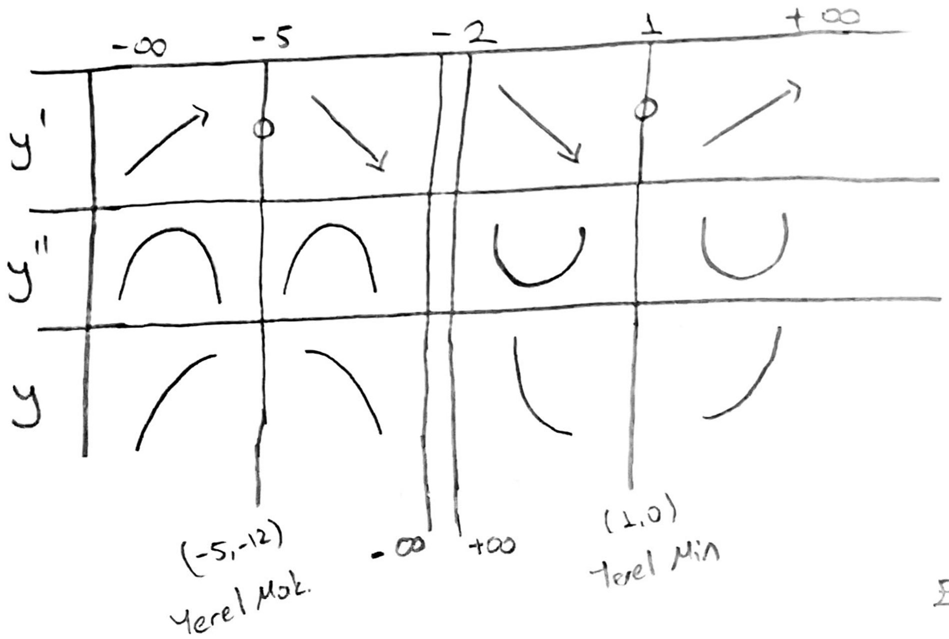
$f''(x) = 0 \Rightarrow \forall x \in T.A$ $\frac{18}{(x+2)^3} \neq 0 \Rightarrow$ Aday büküm noktası yoktur.

f'' tanımsız $\Rightarrow x = -2 \notin T.A$

x		-2	
$f''(x)$	$-$	‖	$+$
$f(x)$	$\cap$	‖	$\cup$

↳ $x = -2$ Büküm noktası $\boxed{\text{değildir.}}$

⑦ Özet Tablo & Grafik

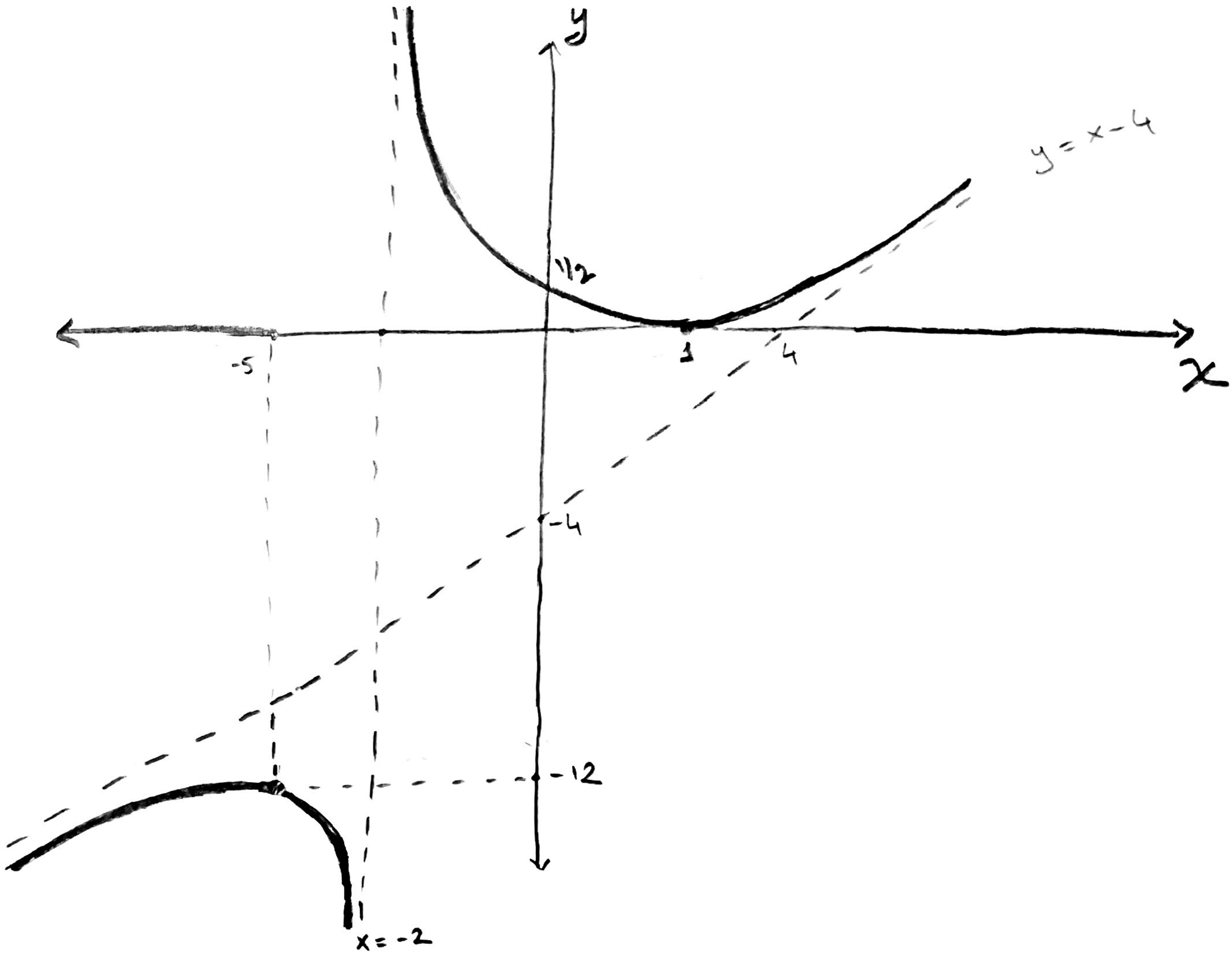


Kesenler:

$(0, \frac{1}{2})$ ve $(1, 0)$

Eğik Asimptot:

$y = x - 4$



Re.6

#1)

$\cos x = 1 - 2\sin^2\frac{x}{2}$

$$\lim_{x\to 0} \frac{1-x^2/8 - 1 + 2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2} - x^2/8}{x^2}$$

$$= \lim_{x\to 0} 2\cdot\left(\frac{\sin x/2}{x}\right)^2 - \lim_{x\to 0}\frac{x^2/8}{x^2} = 2\left(\frac{1}{2}\right)^2 - \frac{1}{8} = 3/8 \text{ //}$$

#2) We will check the continuity of given function at $x=2,4$

• $x=2$

i) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \tan^{-1}\left(\frac{x}{x-2}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

ii) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sin^{-1}\left(\frac{2}{x}\right) = \sin^{-1}(1) = \frac{\pi}{2}$

iii) $f(2) = \frac{\pi}{2}$

So, $f(x)$ is discontinuous at $x=2$.

$f(x)$ has a jump discontinuous at $x=2$ since

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x).$$

• $x=4$

i) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \sin^{-1}\left(\frac{2}{x}\right) = \sin^{-1}\left(\frac{2}{4}\right) = \pi/6$

ii) $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{1}{x-4} = \infty$

So, $f(x)$ is discontinuous at $x=4$, type: infinite discontinuity

#5) Apply d/dx both of sides,

$$2\sin^2(x+y) = 3\sin x + \sin y$$

$$1-\cos(2(x+y)) = 3\sin x + \sin y$$

$$\frac{d}{dx}\left(1-\cos(2(x+y))\right) = \frac{d}{dx}\left(3\sin x + \sin y\right)$$

$$\sin(2(x+y)).2.(1+y') = 3\cos x + \cos y \cdot y' \quad , \quad y' = \frac{dy}{dx}$$

$$\downarrow (x,y) = \left(\frac{\pi}{2}, \frac{\pi}{6}\right) \downarrow$$

$$\sin\left(2\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\right) \cdot 2 \cdot (1+y') = 3\cos\frac{\pi}{2} + \cos\frac{\pi}{6} \cdot y'$$

$$-\frac{\sqrt{3}}{2} . 2(1+y') = 3 \cdot 0 + \frac{\sqrt{3}}{2} \cdot y'$$

$$y' = -\frac{2}{3} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2}, \frac{\pi}{6}\right)} = -\frac{2}{3}$$

Apply d/dx

$$\frac{d}{dx}\left(\sin(2(x+y)).\ 2(1+y')\right) = \frac{d}{dx}\left(3\cos x + \cos y .\ y'\right)$$

$$\cos(2(x+y)) . 4(1+y')^2 + \sin(2(x+y)) . 2 . (y'')$$

$$= -3\sin(x) - \sin y . (y')^2 + \cos y \cdot y''$$

$(x,y) = \left(\frac{\pi}{2}, \frac{\pi}{6}\right)$ and $y' = -\frac{2}{3}$

$$\Rightarrow \cos\left(\frac{4\pi}{3}\right) \cdot 4\left(1-\frac{2}{3}\right)^2 + \sin\left(\frac{4\pi}{3}\right) . 2 . y'' = -3\sin\frac{\pi}{2} - \sin\frac{\pi}{6} . \left(-\frac{2}{3}\right)^2 + \cos\frac{\pi}{6} \cdot y''$$

$$\Rightarrow \quad -\frac{1}{2} \cdot 4 \cdot \frac{1}{9} + \left(-\frac{\sqrt{3}}{2}\right) . 2 . y'' = -3 - \frac{1}{2} \cdot \frac{4}{9} + \frac{\sqrt{3}}{2} . y''$$

$$\left(\frac{\sqrt{3}}{2} + \sqrt{3}\right) y'' = +3 \Rightarrow \frac{3\sqrt{3}}{2} . y'' = 3$$

$$\Rightarrow y'' = \frac{2}{\sqrt{3}} \ //$$