A Case Study on Identifying Bifurcation and Chaos with CROCKER Plots

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İTÜ-Math. & MSU-CMSE

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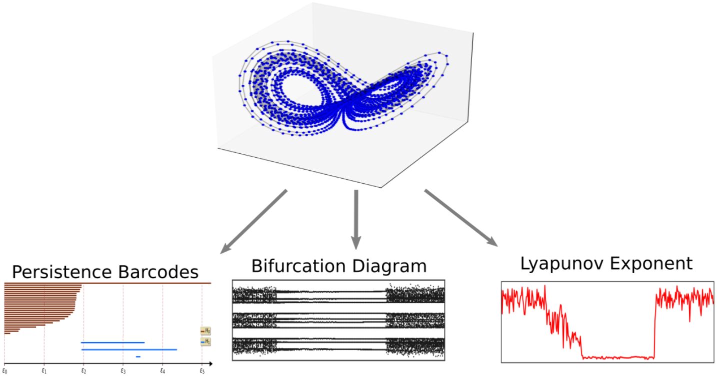
MICHIGAN STATE UNIVERSITY

Outline

1. Dynamical System
   - Bifurcation Diagram
   - Lyapunov Exponent
2. Topological Features
3. CROCKER
   - Norm of CROCKER
4. Experiments
   - Rössler System
   - Lorenz System



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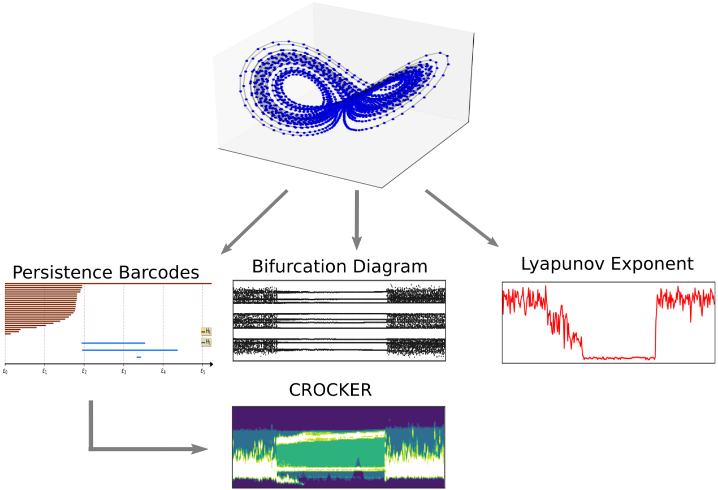

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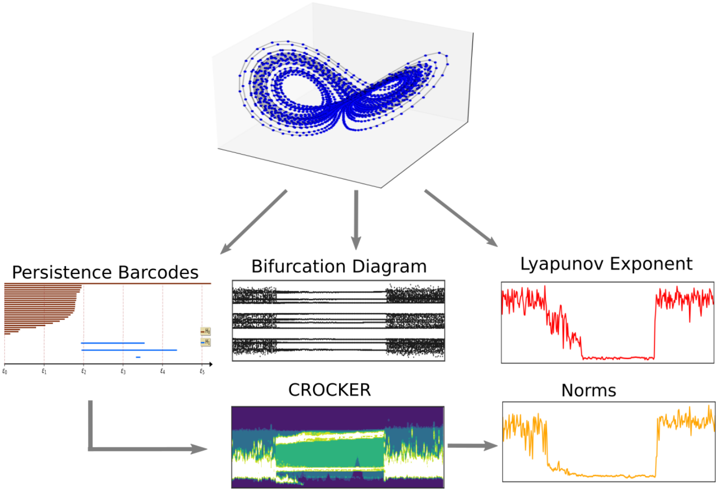


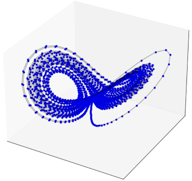
Dynamical System

Dynamical system is a system that changes over time according to a set of fixed rules that determine how one state of the system moves to another state.

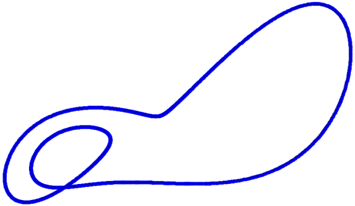
Lorenz system:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(\rho - z) - y, \quad \dot{z} = xy - \beta z$$

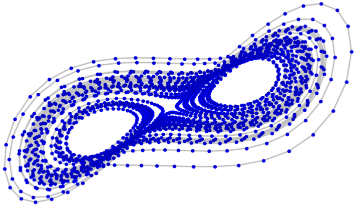
where $\sigma = 10$, $\beta = 8/3$ and $\rho = 105$ with the initial conditions $[0, 0, -1]$.



Periodic



Chaotic



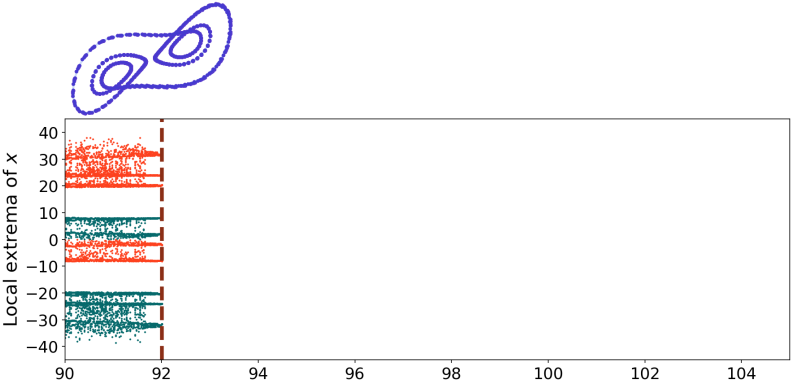
Bifurcation Diagram

Bifurcation diagram is a way to study how a system depends on a parameter.

The Lorenz system is

$$\begin{aligned}\dot{x} =& \sigma(y - x),\\ \dot{y} =& x(\rho - z) - y,\\ \dot{z} =& xy - \beta z,\end{aligned}$$

where the fixed parameters $\sigma = 10$, $\beta = 8/3$ and varying ρ.



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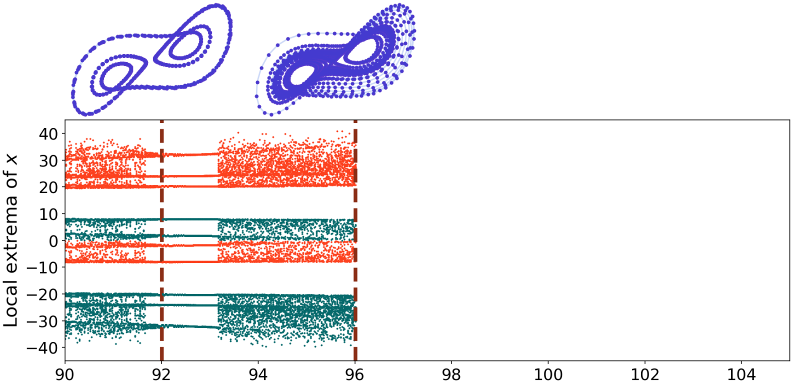

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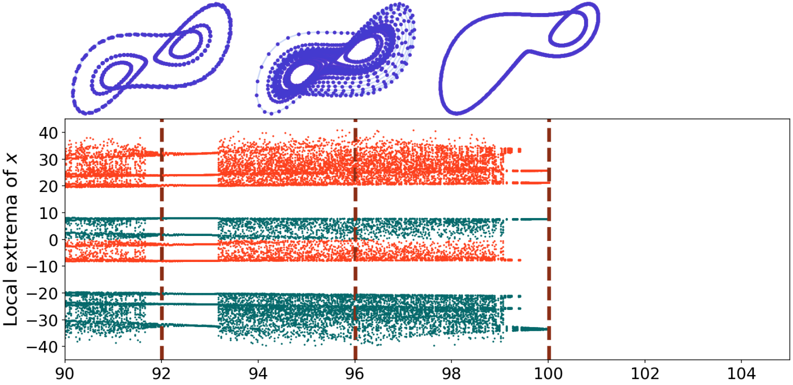

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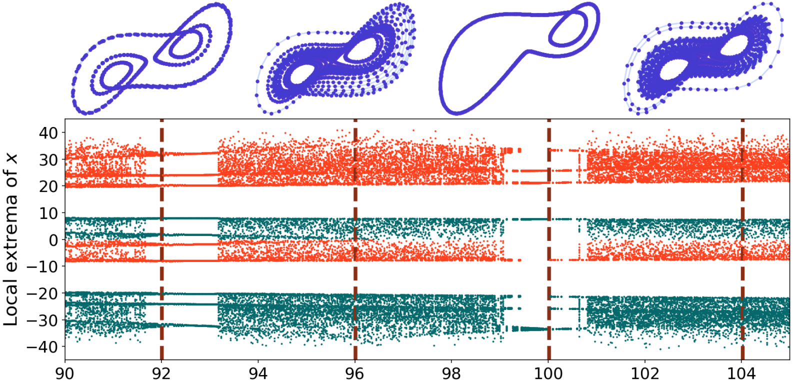
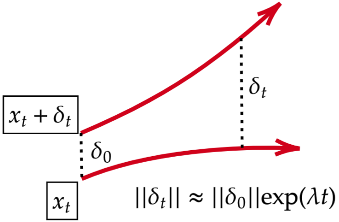
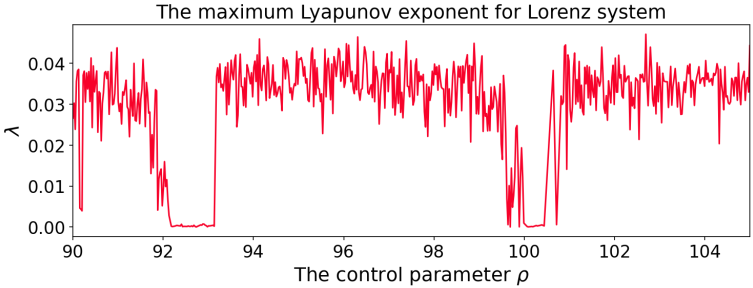


Lyapunov exponent





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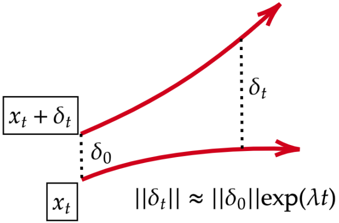
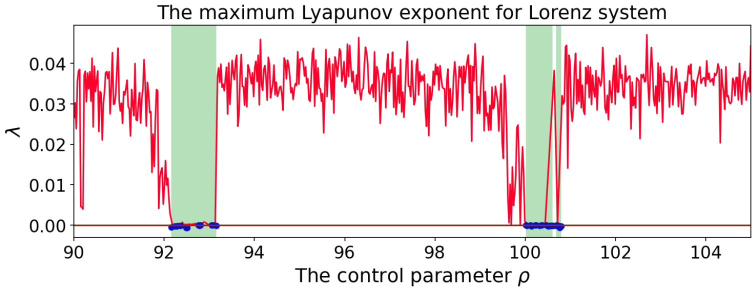


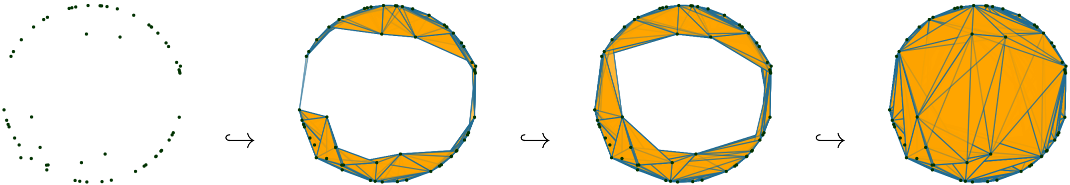


- chaotic if $\lambda > 0$,
- periodic if $\lambda = 0$,
- stable if $\lambda < 0$.

Topological Structure

Given a point cloud X, the Vietoris-Rips is defined to be the simplicial complex whose simplices are built on vertices that are at most ε apart,

$$R_\varepsilon(X) = \{\sigma \subset X \mid d(x, y) \leq \varepsilon, \text{ for all } x, y \in \sigma\}.$$

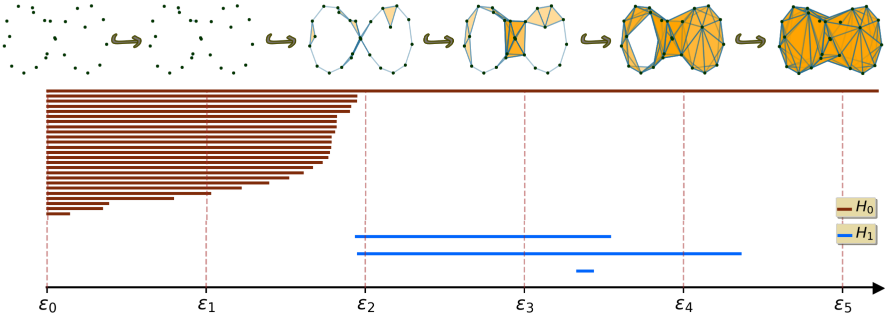


$\beta_0 = 50$	$\hookrightarrow$	$\beta_0 = 1$	$\hookrightarrow$	$\beta_0 = 1$	$\hookrightarrow$	$\beta_0 = 1$
$\beta_1 = 0$		$\beta_1 = 1$		$\beta_1 = 1$		$\beta_1 = 0$

Betti Vector and Persistence Barcode

The p^{th} dimensional Betti vector is defined as

$$Bv_p(X;P) = (\beta_p(R_{\epsilon_0}), \beta_p(R_{\epsilon_1}), \ldots, \beta_p(R_{\epsilon_N}))$$



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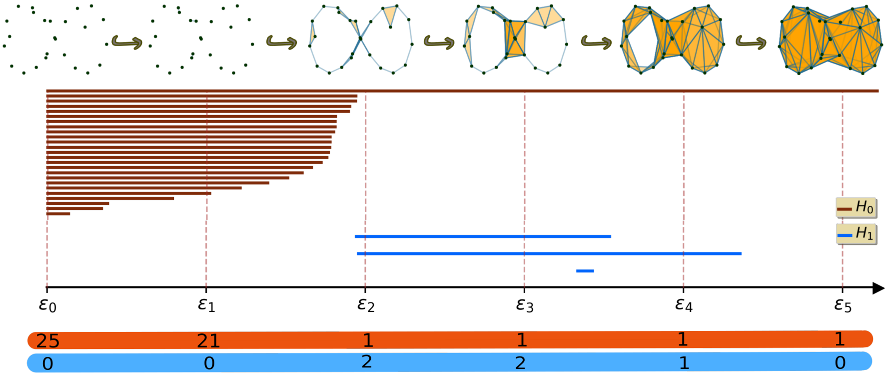

Different But Same

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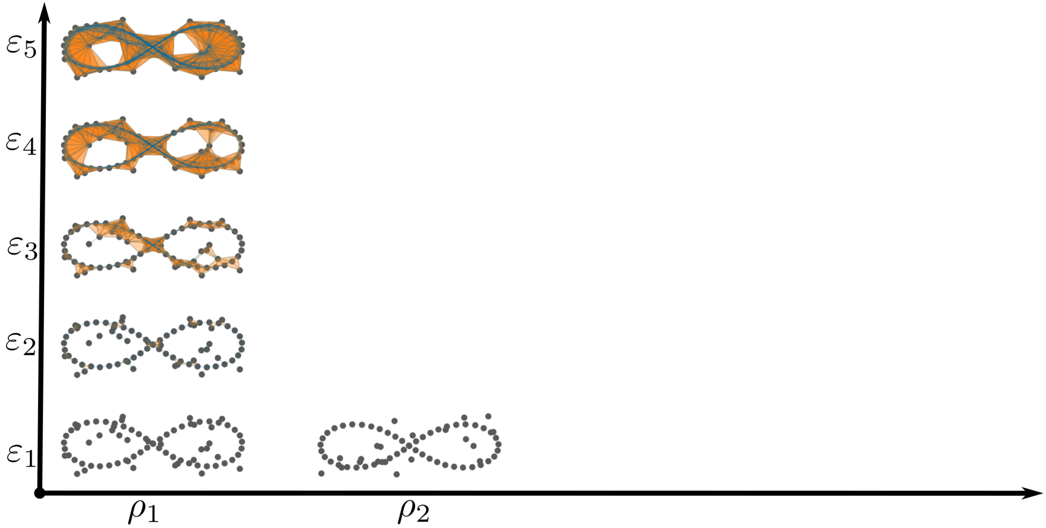

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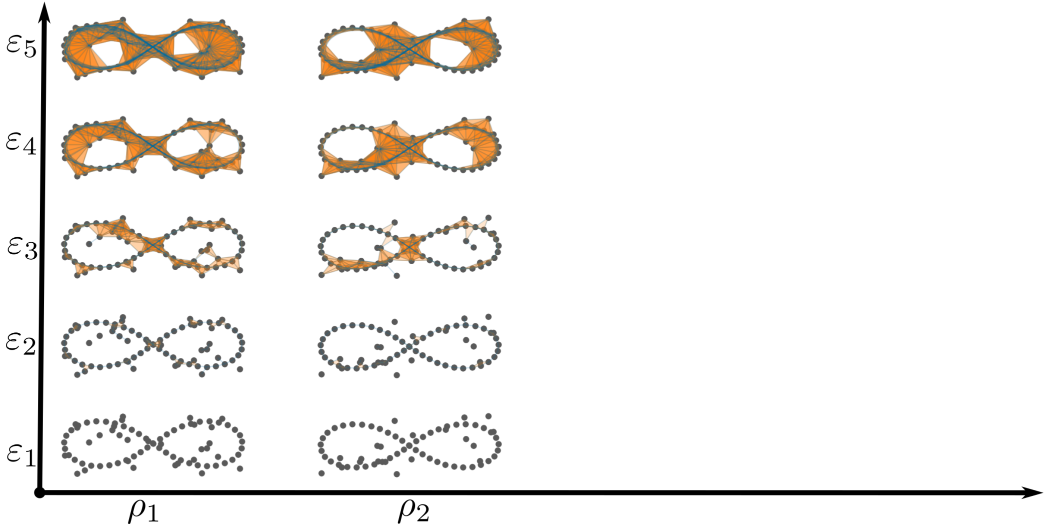

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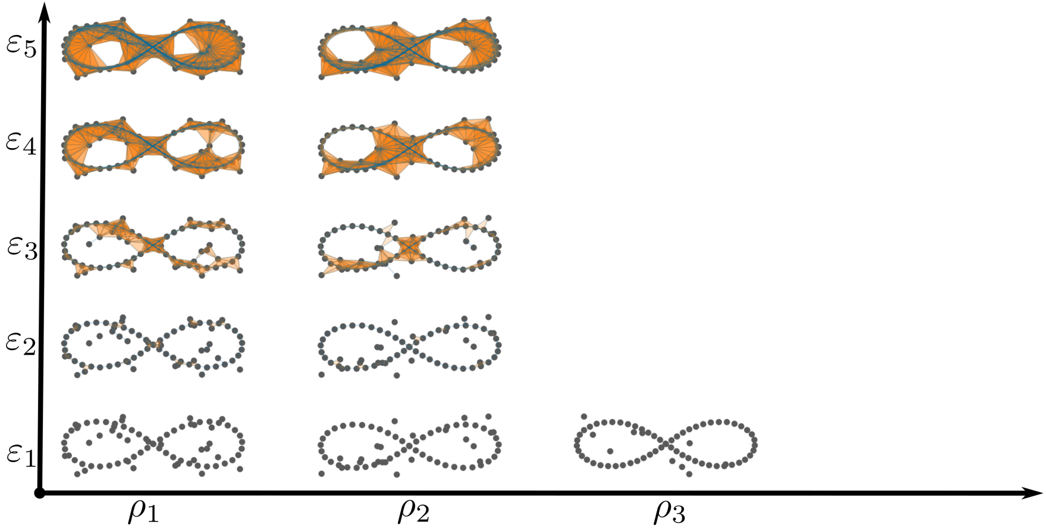

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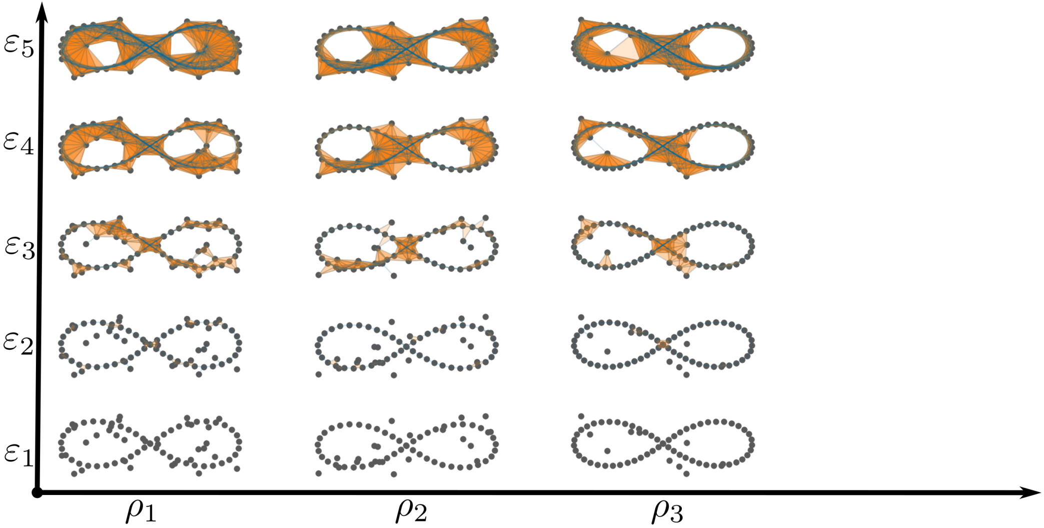

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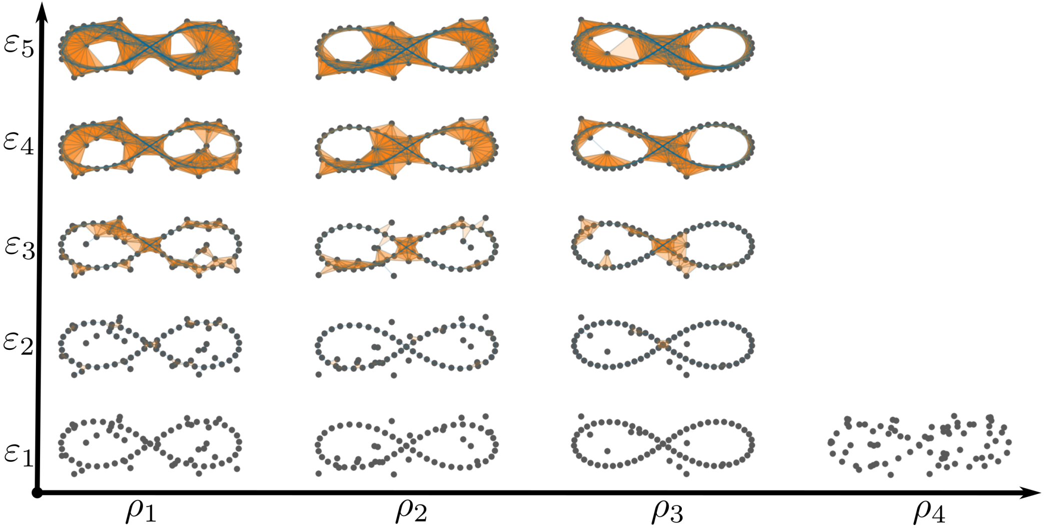

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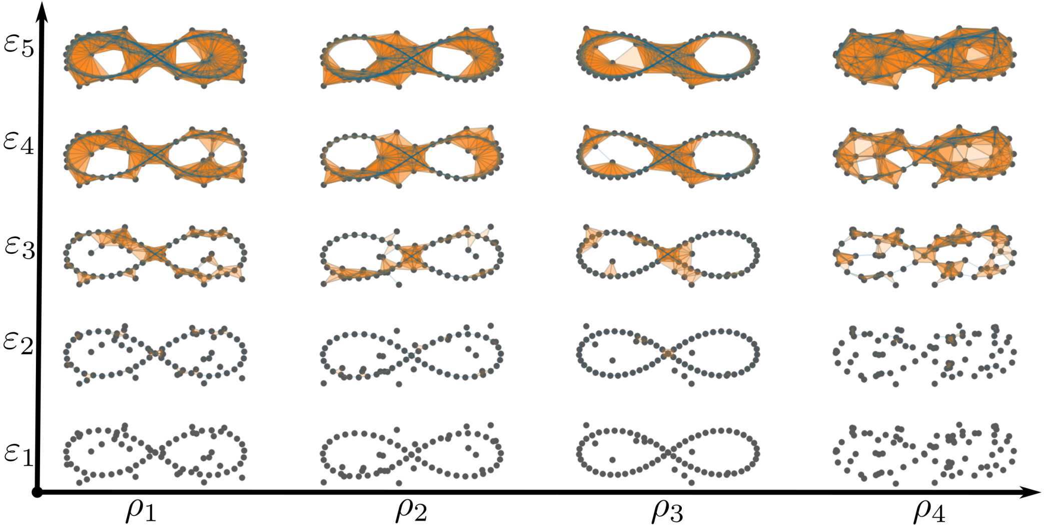

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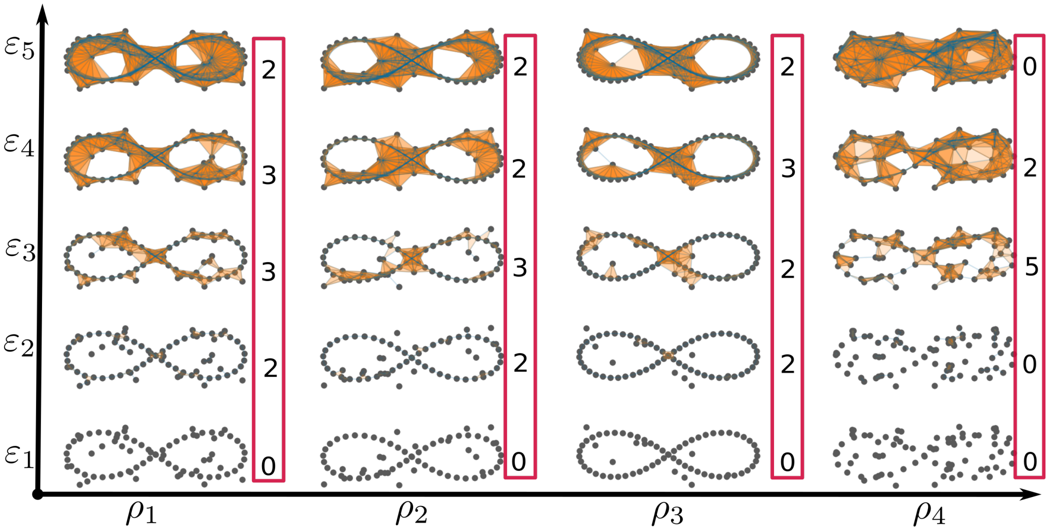


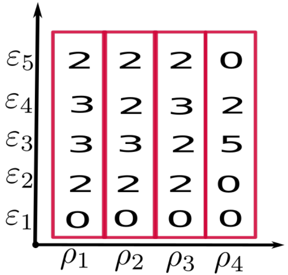
Contour Realization Of Computed k-dimensional hole Evolution in the Rips complex[1]

CROCKER

For a given collection of point clouds $\mathcal{X} = \{X_1, X_2, \cdots, X_T\}$, *CROCKER* of this collection can be given as

$$CROCKER(\mathcal{X}) = (Bv(X_1; P), Bv(X_2; P), \ldots, Bv(X_T; P)),$$

where $Bv(\bullet)$ is the p^{th} dimension Betti vector for the partition $P = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_l\}$.



[1] [Topaz et al., 2015, Ulmer et al., 2019, Bhaskar et al., 2019, Xian et al., 2022]

Algorithm

The Rössler system is

$$\dot{x} = -y - z,$$
$$\dot{y} = x + ay,$$
$$\dot{z} = b + z(x - c),$$

the fixed parameters $b = 2$, $c = 4$ and control parameter a.

1. For each control parameter a,
   - Obtain the states of the nonlinear system.
   - Calculate the full persistence barcode.
2. Find the overall maximum death time d_{max} for each dimension $p \in \{0, 1\}$.
3. Get 100 equally-spaced values of $\varepsilon \in [0, d_{max}]$.
4. For each persistence barcodes,
   - Obtain Betti vectors for each dimension $p \in \{0, 1\}$.
5. Create CROCKER

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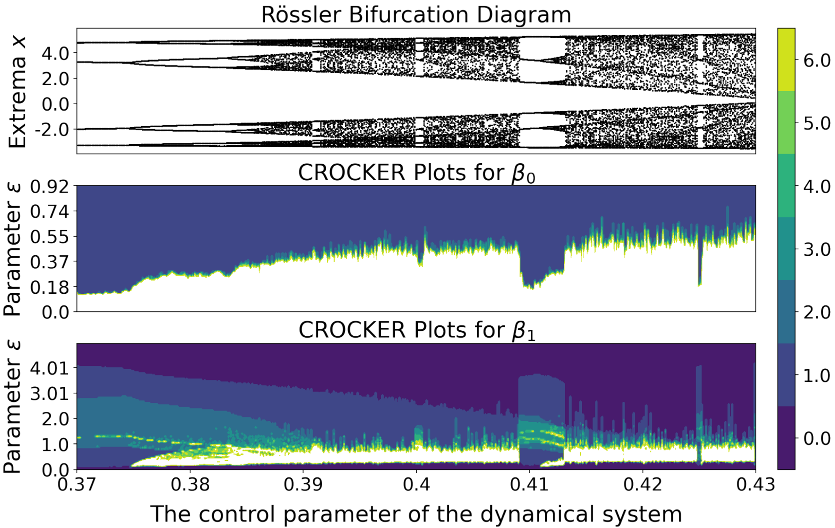

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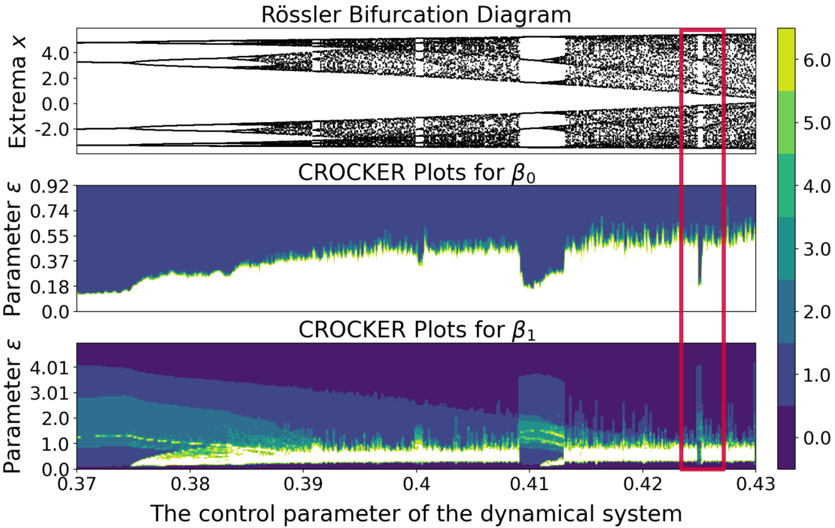

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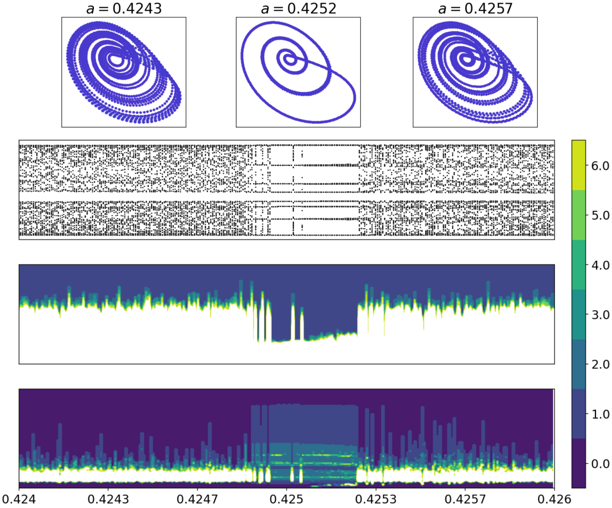
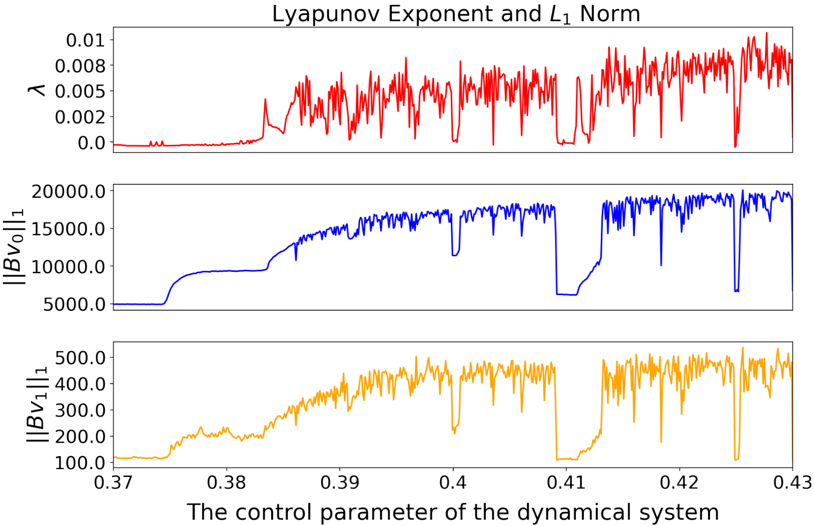
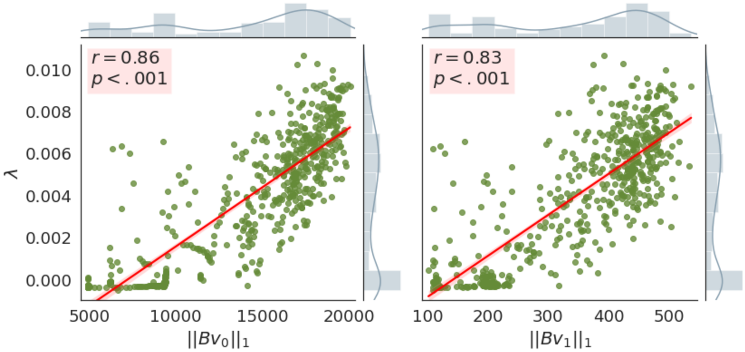


Lyapunov exponent and L_1 norm



Pearson correlation coefficient

	n	r	CI95%	p-val	BF10	power
β_0	600	0.857	[0.83, 0.88]	10^{-173}	10^{169}	1.0
β_1	600	0.832	[0.80, 0.85]	10^{-154}	10^{150}	1.0



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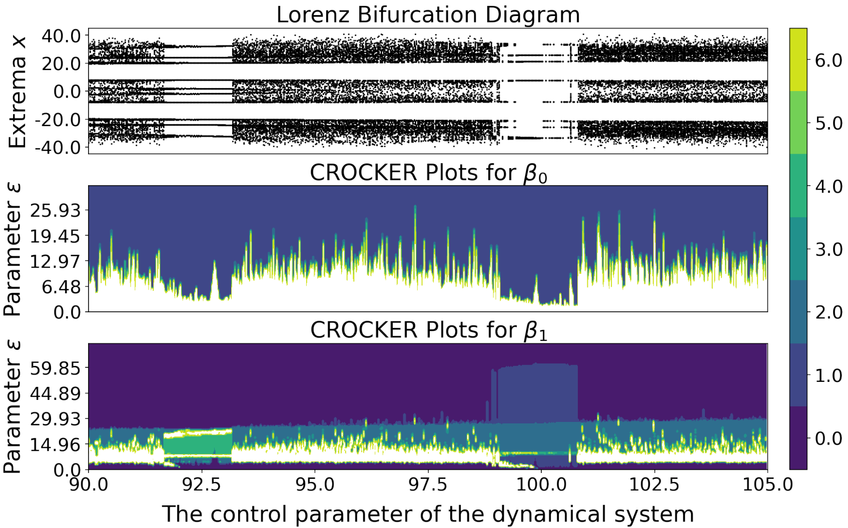

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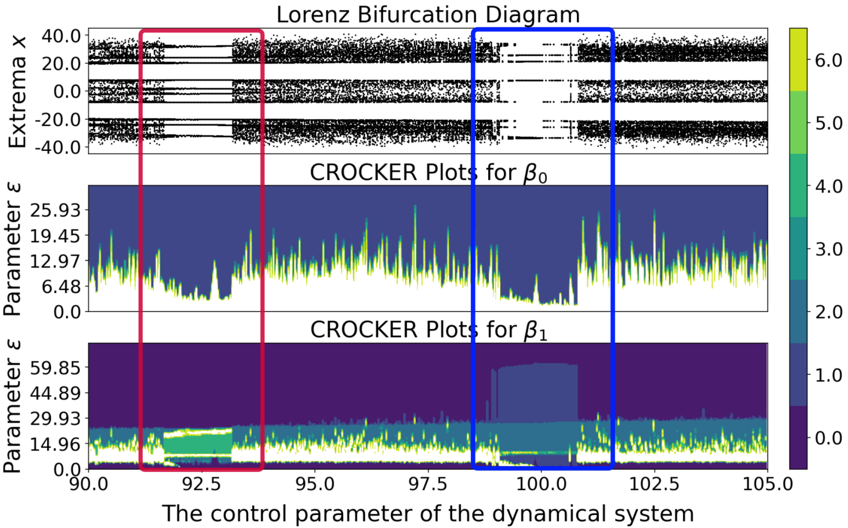

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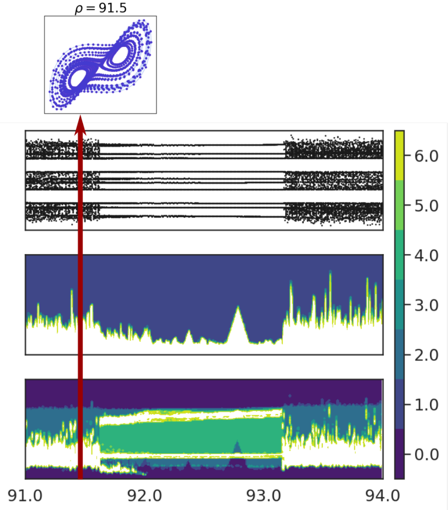

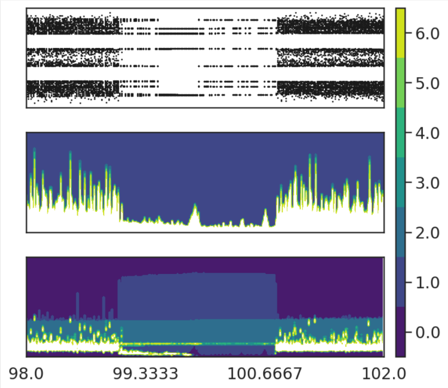

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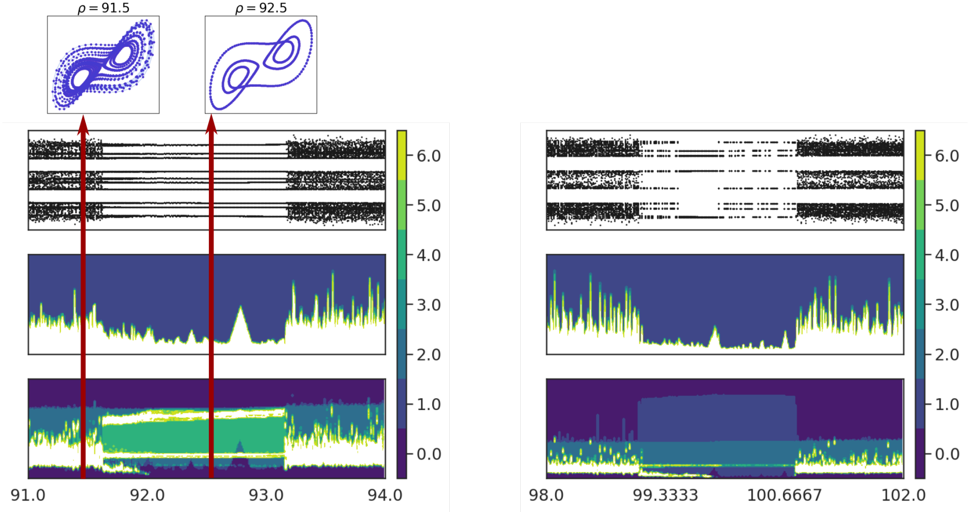

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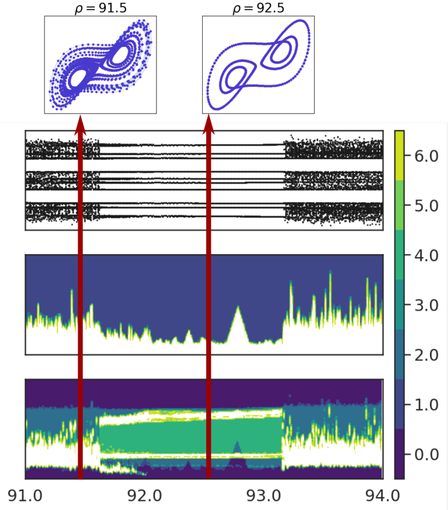

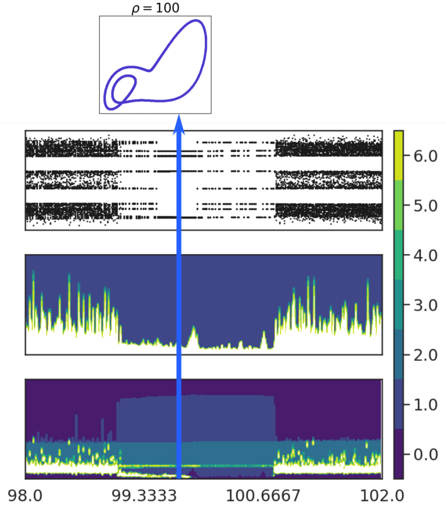

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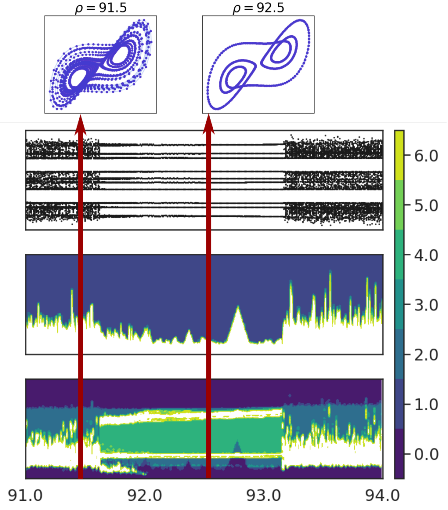

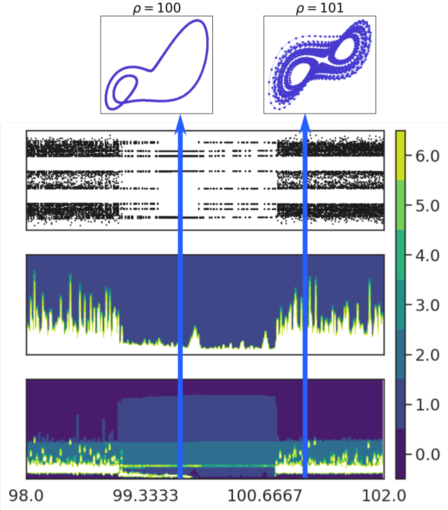

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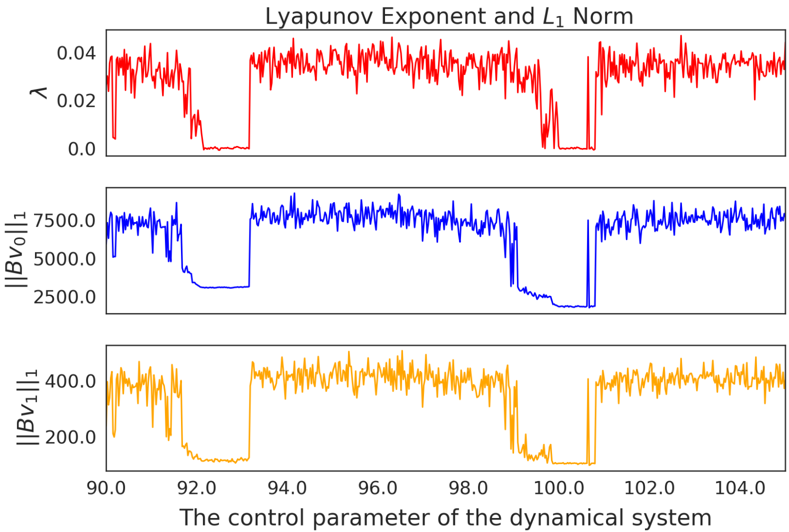

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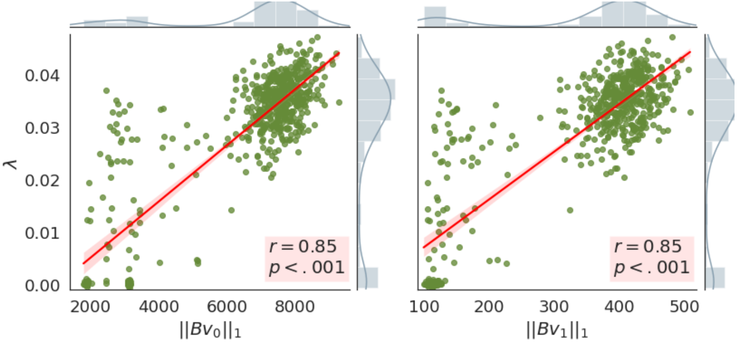

Future Work

- Calculate Betti vectors without full persistence barcodes.
- Nonlinear relation between the Lyapunov exponent and L_1 norms.
- Two or more parameter bifurcations.

References

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Thank You!